Indian Statistical Institute,Bangalore Centre

B.Math.(Hons.)II Year-2013-14,First Semester

Optimization

Final Exam — 13 Nov 2013, 10am - 1pm.

Instructor: P.S.Datti — Max.Marks: 50

NOTE: Solve all the questions. WRITE NEATLY.

1. State the Gram-Schmidt orthogonalization process in $\mathbb{R}^n$. **(2)**

2. Consider the matrix $A = \begin{pmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{pmatrix}$. Obtain the LU decomposition of A and use it to solve the system $Ax = b$, where $b = \begin{pmatrix} 24 \\ 2 \\ 16 \end{pmatrix}$. **(3+3)**

3. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

   (a) Find orthonormal vectors $u^{(1)}, u^{(2)}, u^{(3)}$ in $\mathbb{R}^4$ such that the subspace spanned by them is the subspace spanned by the columns of A. **(5)**

   (b) Find a 4×3 real matrix Q and a 3×3 matrix R such that $Q^tQ = I$ and R is an upper triangular matrix with positive diagonal elements, satisfying $A = QR$. **(3)**

4. Suppose A is a real $m \times n$ matrix of rank n and define $P = A(A^tA)^{-1}A^t$.

   (a) Show that $im(P) = im(A)$ and $ker(P) = ker(A^t)$. **(2)**

   (b) Show that P is an orthogonal projection. **(2)**

5. Suppose A is a non-negative, irreducible matrix of order n. Show that the matrix $(I + A)^{n-1}$ is a positive matrix. **(2)**

6. Let $y_0 = 0$ and $z_0 = 5$. Define for $k = 0, 1, 2, \ldots$

$$\begin{aligned} y_{k+1} &= 0.8y_k + 0.3z_k \\ z_{k+1} &= 0.2y_k + 0.7z_k \end{aligned}$$

   Find the limits of y_k and z_k as $k \to \infty$. **(4)**

7. Reduce the following minimization problem to the standard form the LPP.

$$\text{minimize} \quad |x| + |y| + |z|$$

$$\text{subject to} \quad x + y \leq 1 \text{ and } 2x + z = 3.$$

(2)

8. Consider the following LPP:

$$\text{maximize} \quad c^t x \text{ subject to } Ax \leq b,\ x \geq 0.$$

   (a) Write down the corresponding dual problem. **(1)**

   (b) In the above set up, state and prove the weak duality theorem. **(1+1)**

9. State the Minkowski-Farkas Lemma and prove it using the duality theorem of LP. **(1+2)**

10. (a) Define an extreme point of a convex set in $\mathbb{R}^n$. **(1)**

    (b) Define a feasible solution and a basic feasible solution of a set of constraints $Ax = b,\ x \geq 0$, where A is a real $m \times n$ matrix with rank m. **(1+1)**

    (c) For the above set of constraints, show that a feasible solution is a basic feasible solution if and only if it is an extreme point of the feasible set. **(3+3)**

11. Find a solution of the following LPP using simplex method:

$$\text{minimize} \quad 5x_1 - 8x_2 - 3x_3$$

subject to

$$\begin{aligned} 2x_1 + 5x_2 - x_3 &\leq 1 \\ -3x_1 - 8x_2 + 2x_3 &\leq 4 \\ -2x_1 - 12x_2 + 3x_3 &\leq 9 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

(5)

12. Using the theorem of the alternative, show that the following system

$$\begin{pmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

does not have a non-negative solution. **(2)**